Conformal group and Conformal algebra

A PROJECT REPORT

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Abstract

In this project we start with, some basic definitions in group theory, particularly Lie groups. We then consider the rotational group and the Lorentz group briefly. We then work out the Poincaré algebra and the conformal algebra in some detail. The main theme of this project is to understand, how symmetries like Lorentz and conformal invariance appear in quantum setting.

Introduction

The main theme of this project is to understand, how symmetries like Lorentz and conformal invariance appear in quantum setting? That is, we wish to see how states (and operators) in the Hilbert space of a quantum system transform when we make a Lorentz (or conformal) transformation in coordinate space and what is the meaning of a symmetry transform?

What is a symmetry transformation? Consider an observer $\mathcal{O}$ who finds the state of a system to be $|\psi>$ and another observer $\mathcal{O}'$ who finds the same state to be, $|\psi'>$. Further, consider an orthonormal set state $|\psi_n>$ w.r.t $\mathcal{O}$, which are $|\psi'_n>$ w.r.t $\mathcal{O}'$ then the transformation from $|\psi>$ to $|\psi'>$ is a symmetry transformation if

$$|<\psi_n|\psi>|^2 \;=\; |<\psi'_n|\psi'>|^2$$

Clearly, the condition is satisfied if $|\psi'> = \mathcal{U}|\psi>$ where $\mathcal{U}$ is a unitary operator.*

For symmetry transformations that can be parametrized by continuous parameters and infinitesimally close to identity can be written schematically as

$$\mathcal{U} = 1 + \iota\epsilon t \;(\epsilon\; is\; parameter\; of\; the\; transformation)$$

Since $\mathcal{U}$ is unitary, t is Hermitian. In quantum mechanics hermitian operators are associated with observables. In general, the observables of physics like angular momentum or momentum arise from symmetry transformations. As we will see symmetry transformations of our interest form a group.

Some basic Definitions

Here we quote some definitions and results as in [1]. We also introduce the idea of Lie groups and Lie algebra. There is no attempt at mathematical rigour. This section is only to set the background for the main work of the later sections.

Group: A set G of elements a, b, c, . . . is a group if the following four axioms are satisfied:

1. For every a, b $\in$ {G} , $\exists$ c $\in$ {G}. This operation is indicated as

   $$c = a \circ b$$

   and is called the composition law or the multiplication law of the group.

2. The multiplication is associative, i.e. for any three elements a, b, c $\in$ G:

   $$(a \circ b) \circ c = a \circ (b \circ c)$$

3. The set contains an element e called identity, such that, for each element a $\in$ G,

   $$e \circ a = a \circ e = a$$

*Actually, a fundamental theorem proved by Wigner says that the operator $\mathcal{U}$ is either linear and unitary or anti-linear and anti-unitary. We shall however consider unitary operators only.

4. For all a $\in$ G, $\exists$ a' $\in$ G such that

$$a \circ a' = a' \circ a = e$$

The element a' is called the inverse of a and is denoted by a^{-1}. For example, the set of integers form a group where the operation $a \circ b$ means ordinary addition. The number 0 is the identity and the inverse of a is -a.

Abelian group: If the composition or the multiplication rule is commutative i.e. $a \circ b = b \circ a$ for all a,b $\in$ G then the group is called Abelian group. In general the groups may not be Abelian or commutative. The example above is an Abelian group. Rotation group in three dimensions is a non-Abelian group.

Order of the group: The order of the group is defined as the total no. of elements in the group. It can be finite or infinite, countable or non-countable infinite. Some examples

1. *Finite order:* Symmetric group, the set of n-permutations. This is a non-commutative group and has n! elements.

2. *Non-countable infinite order:* The rotation group is an example. In general all continuous groups are of non-countable infinite order.

Subgroup: A subset H of the group G, of elements a' , b', ... , that is itself a group with the same multiplication law of G, is said to be a subgroup of G. A necessary and sufficient condition for H to be a subgroup of G is that, for any two elements a' ,b' $\in$ H, also $a' \circ b'^{-1} \in$ H. Every group has two trivial subgroups: the group consisting of the identity element alone, and the group itself. A non-trivial subgroup is called a proper subgroup.

Invariant subgroup: Let H be a subgroup of the group G. If for each element h of H, and g of G, the element $g \circ h \circ g^{-1}$ belongs to H, the subgroup H is said to be invariant.

Simple: A group G is said to be simple if it does not contain any invariant subgroups.

Semi-simple: A group G is said to be semi-simple if it does not contain any invariant Abelian subgroups.

Homomorphism: A mapping of a group G onto the group G' is said to be homomorphic if it preserves the composition law. Each element g of G is mapped onto an element g' of G' , which is the image of g, and the product $g_1 \circ g_2$ of two elements of G is mapped onto the product ${g'}_1 \circ {g'}_2$ in G' . In general, the mapping is not one-to-one, several elements of G are mapped onto the same element of G' , but an equal number of elements of G are mapped onto each element of G' . In particular, the unit element e' of G' corresponds to the set of elements $e_1, e_2, \ldots$ of G (only one of these elements coincides with the unit element of G), which is denoted by **E**. The subgroup **E** is an invariant subgroup of G and it is called kernel of the homomorphism.

Isomorphism: If the above map is one-to-one then it is said to be Isomorphic.

Direct product: A group G with two subgroups H_1 and H_2 is said to be a direct product of H_1 and H_2 if:

1. $H_1 \cap H_2 = \mathcal{E}$

2. for all $h_1 \in H_1$ and $h_2 \in H_2$, $[h_1, h_2] = 0$
3. each element g of G is expressible in one and only one way as g $= h_1 \circ h_2$ in terms of the elements h_1 of H_1 and h_2 of H_2.

   The direct product is denoted by G $= H_1 \otimes H_2$.

Semi-direct product: A group G with two subgroups H_1 and H_2 is said to be semi-direct product of H_1 and H_2 if:

1. H_1 is an invariant subgroup of G
2. $H_1 \cap H_2 = \mathcal{E}$
3. each element g of G is expressible in one and only one way as g $= h_1 \circ h_2$ in terms of the elements h_1 of H_1 and h_2 of H_2.

   The semi-direct product is denoted by G $= H_1 \circledS H_2$.

Representation of a group: Let L_n be a n-dimensional complex vector space. If $T : L_n \to L_n$, i.e;

$$x' = T\ x \quad for\ x', x\ \in\ L_n$$

T is a linear operator, i.e;

$$T(\alpha x + \beta y) = \alpha T x + \beta T y$$

If the mapping is one-to-one, then the inverse exists.

If for each element g of G there is a corresponding linear operator $T(g)$ in L_n , such that $T(g_1 \circ g_2) = T(g_1)T(g_2)$. We say that the set of operators $T(g)$ forms a linear (n-dimensional) representation of the group G. The set of operators $T(g)$ is a group G' and in general G is homomorphic to G' . If the mapping of is one-to-one, then G is isomorphic to G'.

Matrix representation: If one fixes a basis in L_n , then the linear transformation performed by the operator T is represented by a $n \times n$ matrix, which is denoted by $D(g)$. The set of matrices $D(g)$ for all $g \in G$ is called n-dimensional matrix representation of the group G. Defining an orthonormal basis $e_1, e_2, \ldots, e_n$ in L_n , the elements of $D(g)$ are given by

$$T(g)e_k = \sum_i D_{ik}(g)e_i$$

and the transformation of a vector x becomes:

$$x'_i = D_{ik}(g)x_k$$

The set of vectors $e_1, e_2, \ldots, e_n$ is called the basis of the representation $D(g)$.

Faithful representation: If the mapping of the group G onto the group of matrices $D(g)$ is one-to-one, the representation $D(g)$ is said to be faithful. Different elements of G correspond to different matrices $D(g)$ and the mapping is isomorphic.

Equivalent representations: If we change the basis of the vector space L_n , the matrices $D(g)$ of a representation are transformed by a non-singular matrix S

$$D'(g) = SD(g)S^{-1}$$

The representations $D(g)$ and $D'(g)$ are said to be equivalent and is called a similarity transform. The two representations are regarded as essentially the same.

Reducible and Irreducible representations: If the representation of operators $T(g)$ of G in L_n leaves a non-trivial subspace L_m of L_n invariant then the representation is called reducible representation. If such a non-trivial invariant subspace does not exist then the representation $T(g)$ is called as irreducible. For reducible representations, it is possible to choose a basis in L_n such that all the matrices corresponding to $T(g)$ can be written in the form

$$D(g) = \begin{pmatrix} D_1(g) & | & D_{12}(g) \\ ------ & | & ------ \\ 0 & | & D_2(g) \end{pmatrix}$$

If L_{n-m} is also invariant then $D(g)$ can be written in the form by a similarity transform

$$D(g) = \begin{pmatrix} D_1(g) & | & 0 \\ ------ & | & ------ \\ 0 & | & D_2(g) \end{pmatrix}$$

and then the representation is completely reducible. Hence, the representation can be written as a direct sum of the two representations.

$$D(g) = D_1(g) \oplus D_2(g)$$

Schur's lemma: It provides a test of irreducibility (for non-Abelian groups) and is stated as

If D(g) is an irreducible representation of the group G, and if

$$AD(g) = D(g)A$$

for all the elements g of G, then A is multiple of the unit matrix.

Unitary representation A representation of the group G is said to be unitary if the matrices $D(g)$, for all the elements g of G, are unitary, i.e.

$$D(g)D(g)^\dagger = D(g)^\dagger D(g) = I$$

where $D(g)^\dagger$ is the adjoint (or the conjugate transposed) or Hermitian conjugate of $D(g)$.

We next introduce Lie groups and Lie algebra. A proper mathematical treatment of this topic is beyond our scope. We will only touch upon some ideas commonly used in physics.

In physics the continuous groups called Lie groups are of great importance. These are groups of transformations $T(\theta)$ with finite set of real continuous parameters, θ^a. Each element of the group is connected to the identity by a path within the group. The group multiplication law is given as

$$T(\bar{\theta})T(\theta) = T(f(\bar{\theta}, \theta))$$

where $f(\bar{\theta}, \theta)$ is a function of the finite set of real parameters. For $\theta^a = 0$ corresponding to the coordinates of the identity we have

$$f^a(\theta, 0) = f^a(0, \theta) = \theta^a$$

Such transformations are represented on the physical Hilbert space by unitary operators $\mathcal{U}(T(\theta))$. For Lie groups these can be represented in the neighbourhood of identity as

$$\mathcal{U}(T(\theta)) = 1 + \iota\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + \dots$$

where t_a,$t_{bc} = t_{cb}$(since, $\theta_b\theta_c$ is symmetric in b and c) are operators independent of $\theta' s$ and t_a are Hermitian. The multiplication law(as stated previously) is

$$\mathcal{U}(T(\bar{\theta}))\mathcal{U}(T(\theta)) = \mathcal{U}(T(f(\bar{\theta}, \theta)))$$

Now we can expand $f^a(\bar{\theta}, \theta)$ as

$$f^a(\bar{\theta}, \theta) = \theta^a + \bar{\theta}^a + f^a{}_{bc}\bar{\theta}^b\theta^c + g^a{}_{bc}\bar{\theta}^b\bar{\theta}^c + h^a{}_{bc}\theta^b\theta^c + \dots$$

But with constraint equation $f^a(\theta, 0) = f^a(0, \theta) = \theta^a$,

$$g^a{}_{bc} = 0 = h^a{}_{bc}$$

Therefore,

$$f^a(\bar{\theta}, \theta) = \theta^a + \bar{\theta}^a + f^a{}_{bc}\bar{\theta}^b\theta^c + \dots$$

Now expanding the multiplication law on both sides as

$$\left[1 + \iota\bar{\theta}^a t_a + \frac{1}{2}\bar{\theta}^b\bar{\theta}^c t_{bc} + \dots\right] \times \left[1 + \iota\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + \dots\right]$$
$$= 1 + \iota\left(\theta^a + \bar{\theta}^a + t_a + f^a{}_{bc}\bar{\theta}^b\theta^c\right)t_a + \frac{1}{2}\left(\theta^b + \bar{\theta}^b + \dots\right)\left(\theta^c + \bar{\theta}^c + \dots\right)t_{bc} + \dots$$

and collecting the coefficients, we get

* $\bar{\theta}^a : \iota t_a - \iota t_a = 0$
* $\bar{\theta}^b\bar{\theta}^c : \frac{1}{2}t_{bc} - \frac{1}{2}t_{bc} = 0$
* $\theta^a : \iota t_a - \iota t_a = 0$
* $\bar{\theta}^\alpha\theta^\beta : \iota^2 t_\alpha t_\beta = \iota f^a{}_{\alpha\beta}t_a + \frac{1}{2}t_{\alpha\beta} + \frac{1}{2}t_{\alpha\beta}$
  $\equiv -t_\alpha t_\beta = \iota f^a{}_{\alpha\beta}t_a + t_{\alpha\beta}$

Since, $t_{\alpha\beta} = t_{\beta\alpha}$ we can write

$$\begin{aligned} t_{\alpha\beta} - t_{\beta\alpha} &= 0 \\ \equiv -t_\alpha t_\beta - \iota f^a{}_{\alpha\beta}t_a + t_\beta t_\alpha + \iota f^a{}_{\beta\alpha}t_a &= 0 \\ \implies [t_\alpha, t_\beta] &= \iota C^a{}_{\alpha\beta}t_a \\ Where\ \ C^a{}_{\alpha\beta} &= -f^a{}_{\alpha\beta} + f^a{}_{\beta\alpha} \end{aligned}$$

The set of real constants $C^a{}_{\alpha\beta}$ are known as structure constants. The operators t_α, t_β called the generators of the group form a Lie algebra under the operation of commutation. Lie algebra $\mathcal{L}$ may be regarded as (finite-dimensional) vector space equipped with a multiplication law denoted by

$$[X, Y] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$$

with the condition

1. $[X,Y] = -[Y,X]$

2. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$
for all $X,Y,Z \in \mathcal{L}$

Finally, we quote some important results about unitary representations of a Lie group.

1. If G is a compact Lie group then every representation of G is equivalent to a unitary representation

2. If G is a compact Lie group then every reducible representation of G is completely reducible

3. If G is a non-compact Lie group then it possesses no finite-dimensional unitary representation other than the trivial representation in which $D(g) = 1$ for all g $\in$ G.

Rotation group

Rotational transformations

The rotational transformations preserve the length of a vector

$$\tilde{x}^i = R^i{}_j \, x^j$$
$$\tilde{x}^2 = x^2 \ (invariance\ condition)$$
$$\implies R^i{}_j \, R^k{}_i = \delta^k{}_j$$

Where $R^i{}_j$ are the elements of the rotational matrix. In the matrix form $RR^T = I$ i.e; R is an orthogonal matrix. The set of orthogonal 3×3 matrices form the orthogonal group $\mathcal{O}(3)$. Taking determinant of the orthogonal condition gives

$$detR = \pm 1$$

The condition $detR = +1$ defines the special orthogonal group, $\mathcal{SO}(3)$. These transformations do not include space inversions and are identified as pure rotations. R is a real orthogonal matrix and has three independent parameters. Hence this is three dimension group. The group $\mathcal{O}(3)$ is neither simple nor semi-simple, whereas, $\mathcal{SO}(3)$ is simple. Also, since the domain of $\mathcal{SO}(3)$ is compact, it is a compact group. $\mathcal{SO}(3)$ is connected as any two points in the parameter space can be connected by a continuous path(to be precise it is doubly connected). Since the group SO(3) is not simply connected, we have to consider its universal covering group, which is the special unitary group $\mathcal{SU}(2)$. The elements of the group $\mathcal{SU}(2)$ are the complex 2×2 unitary matrices with determinant equal to one.

Rotational algebra

The Lie algebra of the rotation group can be realised by considering the infinitesimal transformations of $\mathcal{SO}(3)$ and $\mathcal{SU}(2)$ in a neighbourhood of the unit element. There is a one-to-one correspondence between the infinitesimal transformations of $\mathcal{SO}(3)$ and $\mathcal{SU}(2)$, so that the two groups are locally isomorphic. Therefore, the groups $\mathcal{SO}(3)$ and $\mathcal{SU}(2)$ have the same Lie algebra. The Lie algebra of the $\mathcal{SO}(3)$ group is given by

$$\left[J^i, J^j\right] = \iota \epsilon_{ijk} J^k$$

where J's are the generators of the $\mathcal{SO}(3)$ group. ϵ_{ijk} is the Levi-Civita symbol defined as

$$\begin{aligned}\epsilon_{ijk} &= +1\ (ijk - cyclic)\\ &= -1\ (ijk - acyclic)\\ &= 0\ (otherwise)\end{aligned}$$

Lorentz group

Lorentz transformations

This is an extension of the $\mathcal{SO}(3)$ group but which includes a group of transformations involving time. These transformations are those of boosts that bring one observer from one inertial frame to another related by Lorentz transformation of special relativity. Like in $\mathcal{SO}(3)$, where the transformations preserve the magnitude of a 3-dim vector(distance), the Lorentz transformations preserve the spacetime interval

$$\eta_{\mu\nu} d\tilde{x}^{\mu}\ d\tilde{x}^{\nu} = \eta_{\rho\sigma} dx^{\rho}\ dx^{\sigma} \tag{1}$$

Where $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$.The Lorentz transformations are given as

$$d\tilde{x}^{\mu} = \Lambda^{\mu}{}_{\rho} dx^{\rho} \tag{2}$$

Using equation 2 in 1 we get

$$\eta_{\rho\sigma} = \Lambda^{\mu}{}_{\rho}\ \eta_{\mu\nu}\ \Lambda^{\nu}{}_{\sigma} \tag{3}$$

The relation is really a statement that the metric tensor is invariant under Lorentz transformations from one inertial frame to another. Equation 3 can be written in matrix form as

$$\eta = \Lambda^{T}\ \eta\ \Lambda$$

Lorentz algebra

Taking the determinant of equation 3 and also considering $\rho = 0 = \sigma$ we get

$$det(\Lambda) = \pm 1 \tag{4}$$

$$\Lambda_{00} \geq 1\ or\ \Lambda_{00} \leq 1 \tag{5}$$

These conditions divide the Lorentz group into four subsets. Out of all the four subsets only the one with det(Λ) = +1 (proper) and $\Lambda_{00} \geq 1$ (orthochronous) forms a subgroup because only it contains the identity element, $\mathcal{I}$. Proper Lorentz transformations include rotations and boosts but not space inversions and orthochronous relates to non-reversing of the time flow. This subgroup is called the proper orthchronous Lorentz group or the restricted Lorentz group and is denoted by $\mathcal{SO}^{+}_{\uparrow}$(1,3). Lorentz group describes the set of continuous transformation of boosts and rotations, and is a Lie group. Its infinitesimal transformation can be represented as

$$D = \mathcal{I} + \frac{1}{2}\epsilon_{\mu\nu} M^{\mu\nu} \tag{6}$$

where $M^{\mu\nu}$ generators are defined as the anti-symmetric differential operator, $M^{\mu\nu} = x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}$ and the antisymmetric $\epsilon_{\mu\nu}$ contain the six independent parameters(three for rotations and three for boosts) for the transformation. These generators follow the following commutation relations

$$\iota[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\sigma\mu}M^{\rho\nu} + \eta^{\sigma\nu}M^{\rho\mu} \tag{7}$$

We shall prove these when we go to the more general case of Poincaré algebra. We can identify the rotation and the boost operations in terms of the generators respectively as

$$L_i = -\frac{1}{2}\epsilon_{ijk}M_{jk} \tag{8}$$

$$K_i = M_{0i} \tag{9}$$

In terms of these notations the commutation relations read as

$$[L_i, L_j] = \iota\epsilon_{ijk}L_k \tag{10}$$

$$[K_i, L_j] = \iota\epsilon_{ijk}K_k \tag{11}$$

$$[K_i, K_j] = -\epsilon_{ijk}L_k \tag{12}$$

There is an alternative way of expressing the generators of the Lorentz group. Consider

$$J_i^{\pm} = \frac{1}{2}(L_i \pm \iota K_i) \tag{13}$$

The commutation relations between these are calculated using commutators 10-12

$$\left[J_i^+, J_j^-\right] = \frac{1}{4}([L_i, L_j] - \iota\,[L_i, K_j] + \iota\,[K_i, L_j] - [K_i, K_j])$$

$$= \frac{1}{4}(\iota\epsilon_{ijk}L_k - \epsilon_{ijk}K_k - \epsilon_{ijk}K_k - \iota\epsilon_{ijk}L_k)$$

$$\implies \left[J_i^+, J_j^-\right] = 0 \tag{14}$$

$$\left[J_i^{\pm}, J_j^{\pm}\right] = \frac{1}{4}([L_i, L_j] \pm \iota\,[L_i, K_j] \pm \iota\,[K_i, L_j] - [K_i, K_j])$$

$$= \frac{1}{4}(\iota\epsilon_{ijk}L_k \pm \epsilon_{ijk}K_k \mp \epsilon_{ijk}K_k + \iota\epsilon_{ijk}L_k)$$

$$= \frac{1}{4}(2\iota\epsilon_{ijk}L_k \mp 2\iota\epsilon_{ijk}K_k$$

$$= \frac{1}{2}\iota 2\iota\epsilon_{ijk}(L_k \pm K_k)$$

$$\implies \left[J_i^{\pm}, J_j^{\pm}\right] = \frac{1}{2}\iota\epsilon_{ijk}J_k^{\pm} \tag{15}$$

Equation 14 tells that the two groups (generated by J^+ and J^-) do not overlap as their commutators vanish. Also, each of these forms a group homomorphic to $\mathcal{SO}(3)$, which is apparent from the commutation relation 15. $\mathcal{SO}(3)$ is doubly covered by the group $\mathcal{SU}(2)$ and so by considering only half of the elements as $\mathcal{SU}(2)/\mathcal{Z}_2$, this group can be considered to be isomorphic to $\mathcal{SO}(3)$. Hence,

$$\mathcal{SO}^+_\uparrow(1,3) \simeq \left[\mathcal{SU}(2) \otimes \mathcal{SU}(2)\right]/\mathcal{Z}_2 \tag{16}$$

where each of the $\mathcal{SU}(2)$ groups corresponds to one of the J^+ and J^- generated group elements.

Poincaré group

Ponicaré transformation

$$\tilde{x}^{\mu} = \Lambda^{\mu}{}_{\rho} x^{\rho} + a^{\mu} \tag{17}$$

In relation 17, the first term corresponds to homogeneous Lorentz transformation and the second part is the spacetime translations. Hence, Poincaré transformations are also called inhomogeneous Lorentz transformations.

Consecutive Poincaré transformations

$$x^{\mu} \to \tilde{x}^{\mu} \to \tilde{\tilde{x}}^{\mu}$$

Writing the transformations explicitly

$$\tilde{\tilde{x}}^{\mu} = \bar{\Lambda}^{\mu}{}_{\beta} \tilde{x}^{\beta} + \bar{a}^{\mu}$$

and using equation 17 for $\tilde{x}^{\beta}$ we get

$$\tilde{\tilde{x}}^{\mu} = (\bar{\Lambda}^{\mu}{}_{\beta} \Lambda^{\beta}{}_{\alpha}) x^{\alpha} + (\bar{\Lambda}^{\mu}{}_{\beta} a^{\beta} + \bar{a}^{\mu}) \tag{18}$$

Using equation 1 we can see that this is also a Poincaré transformation. Corresponding to Poincaré transformations on coordinate space, the unitary transformations on vectors in Hilbert space satisfy the following composition rule

$$\mathcal{U}(\bar{\Lambda}, \bar{a}) \mathcal{U}(\Lambda, a) = \mathcal{U}(\bar{\Lambda}\Lambda, \bar{\Lambda} a + \bar{a}) \tag{19}$$

Poincaré algebra

Considering the transformation to be infinitesimal we can expand matrix Λ about Identity and write

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \quad and \quad a^{\mu} = \epsilon^{\mu} \tag{20}$$

Using equation 20 in equation 3 we have;

$$\eta_{\rho\sigma} = \eta_{\mu\nu} (\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma})$$

$$= (\eta_{\rho\nu} + \eta_{\mu\nu} \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma})$$

$$\eta_{\rho\sigma} = \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + O[\omega^2]$$

$$\implies \omega_{\rho\sigma} = -\omega_{\sigma\rho} \quad (anti - symmetric) \tag{21}$$

We consider infinitesimal Lorentz transformation in the Hilbert space:

$$\mathcal{U}(1 + \omega, \epsilon) = \mathcal{I} + \frac{1}{2} \iota \omega_{\rho\sigma} M^{\rho\sigma} - \iota \epsilon_{\rho} P^{\rho} \tag{22}$$

For $\mathcal{U}$ to be unitary we need $\mathcal{U}^{\dagger}\mathcal{U} = \mathcal{I}$. Using expansion 22 in the unitarity relation we get

$$(\mathcal{I} + \frac{1}{2} \iota \omega_{\rho\sigma} M^{\rho\sigma} - \iota \epsilon_{\rho} P^{\rho})^{\dagger} (\mathcal{I} + \frac{1}{2} \iota \omega_{\rho\sigma} M^{\rho\sigma} - \iota \epsilon_{\rho} P^{\rho}) = \mathcal{I}$$

$$\implies (\mathcal{I} - \frac{1}{2} \iota \omega_{\rho\sigma} M^{\rho\sigma\dagger} + \iota \epsilon_{\rho} P^{\rho\dagger})(\mathcal{I} + \frac{1}{2} \iota \omega_{\rho\sigma} M^{\rho\sigma} - \iota \epsilon_{\rho} P^{\rho}) = \mathcal{I}$$

$$\implies \mathcal{I} + \frac{1}{2}\iota\omega_{\rho\sigma}M^{\rho\sigma} - \iota\epsilon_\rho P^\rho - \frac{1}{2}\iota\omega_{\rho\sigma}M^{\rho\sigma\dagger} + \iota\epsilon_\rho P^{\rho\dagger} = \mathcal{I}$$

$$\implies M^{\rho\sigma\dagger} = M^{\rho\sigma} and \;\; P^{\rho\dagger} = P^\rho \tag{23}$$

$$\implies Hermitian\ Operators$$

Since $\omega_{\rho\sigma}$ is anti-symmetric, only the anti-symmetric part in $M^{\rho\sigma}$ would contribute.

$$\implies M^{\rho\sigma} = -M^{\sigma\rho} \tag{24}$$

Transformation properties of $M^{\rho\sigma}$ and P^ρ

$$Eq\ 19\ gives\ \ \mathcal{U}^{-1}(\Lambda, a) = \mathcal{U}(\Lambda^{-1}, -\Lambda^{-1}a) \tag{25}$$

Now using equation 25

$$\mathcal{U}(\Lambda, a)\mathcal{U}(1+\omega, \epsilon)\mathcal{U}^{-1}(\Lambda, a) = \mathcal{U}(\Lambda, a)\mathcal{U}(1+\omega, \epsilon)\mathcal{U}(\Lambda^{-1}, -\Lambda^{-1}a)$$

Using the composition rule i.e; equation 19 we have

$$\mathcal{U}(\Lambda, a)\mathcal{U}(1+\omega, \epsilon)\mathcal{U}(\Lambda^{-1}, -\Lambda^{-1}a) = \mathcal{U}(\Lambda, a)\mathcal{U}((1+\omega)\Lambda^{-1}, (1+\omega)(-\Lambda^{-1}a) + \epsilon)$$

$$= \mathcal{U}(\Lambda(1+\omega)\Lambda^{-1}, \Lambda((1+\omega)(-\Lambda^{-1}a) + \epsilon) + a)$$

$$\implies \mathcal{U}(\Lambda, a)\mathcal{U}(1+\omega, \epsilon)\mathcal{U}^{-1}(\Lambda, a) = \mathcal{U}(\Lambda(1+\omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a) \tag{26}$$

Using expansion 22 in relation 26 we have upto first order in ω and ϵ

$$\mathcal{U}(\Lambda, a)(\mathcal{I} + \frac{1}{2}\iota\omega_{\rho\sigma}M^{\rho\sigma} - \iota\epsilon_\rho P^\rho)\mathcal{U}^{-1}(\Lambda, a) = \mathcal{U}(\Lambda(1+\omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)$$

Writing an expansion of r.h.s in above as in expansion 22 we have

$$\mathcal{U}(\Lambda, a)(\mathcal{I} + \frac{1}{2}\iota\omega_{\rho\sigma}M^{\rho\sigma} - \iota\epsilon_\rho P^\rho)\mathcal{U}^{-1}(\Lambda, a) = \mathcal{I} + \frac{1}{2}\iota(\Lambda\omega\Lambda^{-1})_{\mu\nu}M^{\mu\nu} - \iota(\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)_\mu P^\mu$$

$$= \mathcal{I} + \frac{1}{2}\iota\Lambda_\mu{}^\rho\omega_{\rho\sigma}(\Lambda^{-1})^\sigma{}_\nu M^{\mu\nu} - \iota(\Lambda_\mu{}^\rho\epsilon_\rho - \Lambda_\mu{}^\rho\omega_{\rho\sigma}(\Lambda^{-1})^\sigma{}_\nu a^\nu)P^\mu$$

Relation between Λ^{-1} and Λ

$$(\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho \equiv \eta_{\nu\mu}\eta^{\rho\sigma}\Lambda^\mu{}_\sigma \tag{27}$$

$$\implies \mathcal{U}(\Lambda, a)(\mathcal{I} + \frac{1}{2}\iota\omega_{\rho\sigma}M^{\rho\sigma} - \iota\epsilon_\rho P^\rho)\mathcal{U}^{-1}(\Lambda, a) = \mathcal{I} + \frac{1}{2}\iota\Lambda_\mu{}^\rho\omega_{\rho\sigma}\Lambda_\nu{}^\sigma M^{\mu\nu} - \iota(\Lambda_\mu{}^\rho\epsilon_\rho - \Lambda_\mu{}^\rho\omega_{\rho\sigma}\Lambda_\nu{}^\sigma a^\nu)P^\mu$$

$$\implies \mathcal{U}(\Lambda, a)\mathcal{I}\mathcal{U}^{-1}(\Lambda, a) + \frac{1}{2}\iota\omega_{\rho\sigma}\mathcal{U}(\Lambda, a)M^{\rho\sigma}\mathcal{U}^{-1}(\Lambda, a) - \iota\epsilon_\rho\mathcal{U}(\Lambda, a)P^\rho\mathcal{U}^{-1}(\Lambda, a)$$

$$= \mathcal{I} + (\frac{1}{2}\iota M^{\mu\nu} + \iota a^\nu P^\mu)\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\omega_{\rho\sigma} - \iota(\Lambda_\mu{}^\rho P^\mu)\epsilon_\rho$$

Equating the coefficients of $\omega_{\rho\sigma}$ and ϵ_ρ on both side and considering only the anti-symmetric part of $\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma a^\nu P^\mu$, since only it contributes when multiplied by $\omega_{\rho\sigma}$(anti-symmetric)

$$\mathcal{U}(\Lambda, a)M^{\rho\sigma}\mathcal{U}^{-1}(\Lambda, a) = \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma(M^{\mu\nu} + a^\nu P^\mu - a^\mu P^\nu) \tag{28}$$

$$\mathcal{U}(\Lambda, a)P^\rho\mathcal{U}^{-1}(\Lambda, a) = \Lambda_\mu{}^\rho P^\mu \tag{29}$$

Consider (Λ, a) to be infinitesimal transformation as in expansion 22. Then the L.H.S from relation 28 can be written as

$$\mathcal{U}(1+\omega,\epsilon)M^{\rho\sigma}\mathcal{U}^{-1}(1+\omega,\epsilon) = (\mathcal{I} + \frac{1}{2}\iota\omega_{\mu\nu}M^{\mu\nu} - \iota\epsilon_\mu P^\mu)M^{\rho\sigma}(\mathcal{I} - \frac{1}{2}\iota\omega_{\mu\nu}M^{\mu\nu} + \iota\epsilon_\mu P^\mu)$$

Upto first order in ω and ϵ we can write

$$= M^{\rho\sigma} - \frac{1}{2}\iota M^{\rho\sigma}\omega_{\mu\nu}M^{\mu\nu} + \iota M^{\rho\sigma}\epsilon_\mu P^\mu + \frac{1}{2}\iota\omega_{\mu\nu}M^{\mu\nu}M^{\rho\sigma} - \iota\epsilon_\mu P^\mu M^{\rho\sigma}$$

$$= M^{\rho\sigma} - \iota(\frac{1}{2}M^{\rho\sigma}\omega_{\mu\nu}M^{\mu\nu} - M^{\rho\sigma}\epsilon_\mu P^\mu) + \iota(\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}M^{\rho\sigma} - \epsilon_\mu P^\mu M^{\rho\sigma})$$

$$\implies \mathcal{U}(1+\omega,\epsilon)M^{\rho\sigma}\mathcal{U}^{-1}(1+\omega,\epsilon) = M^{\rho\sigma} + \iota\left[\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu} - \epsilon_\mu P^\mu, M^{\rho\sigma}\right] \tag{30}$$

R.H.S from relation 28 can be written as, using expansion 20

$$\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma(M^{\mu\nu} + a^\nu P^\mu - a^\mu P^\nu) = (\delta_\mu{}^\rho + \omega_\mu{}^\rho)(\delta_\nu{}^\sigma + \omega_\nu{}^\sigma)(M^{\mu\nu} + \epsilon^\nu P^\mu - \epsilon^\mu P^\nu)$$

$$= (\delta_\mu{}^\rho\delta_\nu{}^\sigma + \delta_\mu{}^\rho\omega_\nu{}^\sigma) + \omega_\mu{}^\rho\delta_\nu{}^\sigma + O[\omega^2])(M^{\mu\nu} + a^\nu P^\mu - a^\mu P^\nu)$$

$$\implies \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma(M^{\mu\nu} + a^\nu P^\mu - a^\mu P^\nu) = M^{\rho\sigma} - \epsilon^\rho P^\sigma + \epsilon^\sigma P^\rho + \omega_\nu{}^\sigma M^{\rho\nu} + \omega_\mu{}^\rho M^{\mu\sigma} \tag{31}$$

From equations 28, 30 and 31 we can write

$$\iota\left[\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu} - \epsilon_\mu P^\mu, M^{\rho\sigma}\right] = \omega_\nu{}^\sigma M^{\rho\nu} + \omega_\mu{}^\rho M^{\mu\sigma} - \epsilon^\rho P^\sigma + \epsilon^\sigma P^\rho \tag{32}$$

Following on the same line of arguments and using equation 29 we have

$$\iota\left[\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu} - \epsilon_\mu P^\mu, P^\rho\right] = \omega_\mu{}^\rho P^\mu \tag{33}$$

We can write equation 32 as

$$\iota\frac{1}{2}\left(\omega_{\mu\nu}\left[M^{\mu\nu}, M^{\rho\sigma}\right] - \epsilon_\mu\left[P^\mu, M^{\rho\sigma}\right]\right) = \omega_{\nu\mu}\eta^{\mu\sigma}M^{\rho\nu} + \omega_{\mu\nu}\eta^{\nu\rho}M^{\mu\sigma} - \epsilon_\mu\eta^{\mu\rho}P^\sigma + \epsilon_\mu\eta^{\mu\sigma}P^\rho$$

$$\implies \iota\frac{1}{2}\omega_{\mu\nu}\left[M^{\mu\nu}, M^{\rho\sigma}\right] - \iota\epsilon_\mu\left[P^\mu, M^{\rho\sigma}\right] = \omega_{\mu\nu}\left(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\sigma}M^{\rho\nu}\right) - \epsilon_\mu\left(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho\right) \tag{32'}$$

Since, $\omega_{\mu\nu}$ is anti-symmetric, only the anti-symmetric coefficient of $\omega_{\mu\nu}$ in the R.H.S contributes. Equating the coefficients of $\omega_{\mu\nu}$ and ϵ_μ on both sides of (32') and (33), we get

$$\iota\left[M^{\mu\nu}, M^{\rho\sigma}\right] = \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\sigma}M^{\rho\nu} - \eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\rho\mu} \tag{34}$$

$$\iota\left[P^\mu, M^{\rho\sigma}\right] = \eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho \tag{35}$$

$$\left[P^\mu, P^\rho\right] = 0 \tag{36}$$

As expected, we get the commutation relations (7) corresponding to boosts and rotations (homogeneous Lorentz transformations) and two other commutators involving translation operation. These commutators form the Lie algebra of the Poincaré group.

Conserved Operators

In commutator 36 identifying $P^0 = H$ and $P^i,\ i \in \{1,2,3\}$ we can write

$$\left[P^0, P^i\right] \equiv \left[H, P^i\right] = 0 \tag{37}$$

$$\implies \textit{these are conserved operators}$$

Similarly from commutator 35 we can write

$$\left[P^0, M^{\rho\sigma}\right] = \eta^{0\rho}P^\sigma - \eta^{0\sigma}P^\rho$$

Now for $\rho, \sigma \neq 0$ we can write

$$\left[P^0, M^{\rho\sigma}\right] = 0 \tag{38}$$

$$\implies \{M^{12}, M^{23}, M^{31}\}\ \textit{commute with H and hence are conserved operators,}$$

$$\textit{identified as angular} - \textit{momentum operators}$$

For ρ *or* $\sigma = 0$ we see that $\{M^{01}, M^{02}, M^{03}\} \equiv K$'s do not commute with H. These are the boosts. Using relations 8 and 9 and also writing $P^0 = H$ we can write the Lie algebra in three dimensional notations as

$$[L_i, L_j] = \iota\epsilon_{ijk}L_k \tag{39}$$

$$[K_i, L_j] = \iota\epsilon_{ijk}K_k \tag{40}$$

$$[K_i, K_j] = -\epsilon_{ijk}L_k \tag{41}$$

$$[L_i, P_j] = \iota\epsilon_{ijk}P_k \tag{42}$$

$$[K_i, P_j] = -\iota H\delta_{ij} \tag{43}$$

$$[K_i, H] = -\iota P_i \tag{44}$$

$$[P_i, P_j] = 0 \tag{45}$$

$$[L_i, H] = 0 \tag{46}$$

$$[P_i, H] = 0 \tag{47}$$

$$[H, H] = 0 \tag{48}$$

We see that Poincaré algebra is a ten-dimensional Lie algebra, with ten generators(three for rotations, three for boosts and four for spacetime translations).

Conformal group

Conformal transformations

A conformal group is a group of (non-linear) coordinate transformations which leaves the metric invariant upto a scale. That is, referring to relation 1,

$$\eta_{\mu\nu}d\tilde{x}^\mu\ d\tilde{x}^\nu = \mathcal{S}(x)\ \eta_{\rho\sigma}dx^\rho\ dx^\sigma \tag{49}$$

$$i.e;\quad \eta_{\mu\nu}\ \partial_\rho\tilde{x}^\mu\ \partial_\sigma\tilde{x}^\nu dx^\rho\ dx^\sigma = \mathcal{S}(x)\ \eta_{\rho\sigma}dx^\rho\ dx^\sigma$$

$$\implies \eta_{\mu\nu}\ \partial_\rho\tilde{x}^\mu\ \partial_\sigma\tilde{x}^\nu = \mathcal{S}(x)\ \eta_{\rho\sigma} \tag{50}$$

We see that for $\mathcal{S}(x) = 1$ in equation 49 gives back the Lorentz condition, equation 1. Therefore, Poincaré transformations are a special case of conformal transformations.

Next we consider a general infinitesimal coordinate transformation as:

$$\tilde{x}^a = x^a + f^a(x) \tag{51}$$

Using this transformation in equation 50 we can write

$$\eta_{\mu\nu}\ \frac{\partial}{\partial x^\rho}(x^\mu + f^\mu)\frac{\partial}{\partial x^\sigma}(x^\nu + f^\nu) = \mathcal{S}(x)\ \eta_{\rho\sigma}$$

$$\implies \eta_{\mu\nu}\ (\delta^\mu{}_\rho + \frac{\partial f^\mu}{\partial x^\rho})(\delta^\nu{}_\sigma + \frac{\partial f^\nu}{\partial x^\sigma}) = \mathcal{S}(x)\ \eta_{\rho\sigma}$$

$$i.e;\quad \eta_{\mu\nu}\ (\delta^\mu{}_\rho\delta^\nu{}_\sigma + \frac{\partial f^\mu}{\partial x^\rho}\delta^\nu{}_\sigma + \delta^\mu{}_\rho\frac{\partial f^\nu}{\partial x^\sigma}) = \mathcal{S}(x)\ \eta_{\rho\sigma}$$

$$\implies \partial_\rho f_\sigma + \partial_\sigma f_\rho = (\mathcal{S}(x) - 1)\ \eta_{\rho\sigma} \tag{52}$$

Now taking the trace of equation 52 we get

$$\eta^{\rho\sigma}\,(\partial_\rho f_\sigma + \partial_\sigma f_\rho) = (\mathcal{S}(x) - 1)\ \eta^{\rho\sigma}\eta_{\rho\sigma} \equiv \Lambda(x)\eta^{\rho\sigma}\eta_{\rho\sigma}$$

$$(\partial^\sigma f_\sigma + \partial^\rho f_\rho) = (\mathcal{S}(x) - 1)\ n \equiv \Lambda(x)n$$

$$\implies 2(\partial \cdot f) = n\Lambda(x) \tag{53}$$

Using equation 53 we can write equation 52 as

$$\implies \partial_\rho f_\sigma + \partial_\sigma f_\rho = \frac{2}{n}(\partial \cdot f)\ \eta_{\rho\sigma} \tag{54}$$

Finding the form of f^a

Applying a derivative on equation 52 as

$$\partial_c\,(\partial_\rho f_\sigma + \partial_\sigma f_\rho) = \eta_{\rho\sigma}\partial_c\Lambda$$

and taking a cyclic permutation $(c\rho\sigma) \to (\rho\sigma c)$ we can write above as

$$\partial_\rho\,(\partial_\sigma f_c + \partial_c f_\sigma) = \eta_{\sigma c}\partial_\rho\Lambda \tag{55}$$

again doing $(\rho\sigma c) \to (\sigma c\rho)$ we can write equation 55 as

$$\partial_\sigma\,(\partial_c f_\rho + \partial_\rho f_c) = \eta_{c\rho}\partial_\sigma\Lambda \tag{56}$$

Now subtracting equation 56 from equation 55 we have

$$\partial_\rho\,(\partial_\sigma f_c + \partial_c f_\sigma) - \partial_\sigma\,(\partial_c f_\rho + \partial_\rho f_c) = \eta_{\sigma c}\partial_\rho\Lambda - \eta_{c\rho}\partial_\sigma\Lambda$$

$$\implies \partial_c\,(\partial_\rho f_\sigma - \partial_\sigma f_\rho) = \eta_{\sigma c}\partial_\rho\Lambda - \eta_{c\rho}\partial_\sigma\Lambda \tag{57}$$

Integrating equation 57

$$\int \partial_c\,(\partial_\rho f_\sigma - \partial_\sigma f_\rho)\,dx^c = \int (\eta_{\sigma c}\partial_\rho\Lambda - \eta_{c\rho}\partial_\sigma\Lambda)\,dx^c$$

$$\implies \partial_\rho f_\sigma - \partial_\sigma f_\rho = \int (\partial_\rho\Lambda\ dx_\sigma - \partial_\sigma\Lambda\ dx_\rho) + 2\omega_{\rho\sigma} \tag{58}$$

Where $\omega_{\rho\sigma}$ is an anti-symmetric constant tensor, since L.H.S is anti-symmetric in ρ and σ. Now adding equation's 52 and 58 we can write

$$2\ \partial_\rho f_\sigma = \eta_{\rho\sigma}\Lambda + \int (\partial_\rho\Lambda\ dx_\sigma - \partial_\sigma\Lambda\ dx_\rho) + 2\omega_{\rho\sigma} \tag{59}$$

Integrating equation 59, we can write

$$\int dx^\rho \partial_\rho f_\sigma = f_\sigma = \frac{1}{2}\int \eta_{\rho\sigma}\Lambda\ dx^\rho + \frac{1}{2}\int dx^\rho \int (\partial_\rho\Lambda\ dx_\sigma - \partial_\sigma\Lambda\ dx_\rho) + \int dx^\rho \omega_{\rho\sigma} + a_\sigma$$

$$Thus, \quad f^a = a^a + \omega^{ba}x_b + \frac{1}{2}\int\ dx^a\Lambda + \frac{1}{2}\int dx^b \int (\partial_b\Lambda\ dx^a - \partial^a\Lambda\ dx_b) \tag{60}$$

Once the function Λ is found, the conformal killing vector f can be determined from equation 60. So, next we find the function Λ. But before that, we see that the first two terms in equation 60 represent the Poincaré transformations. This is the case when $\Lambda = 0 \equiv \mathcal{S} = 1$, which corresponds to coordinate transformations that do not change the form of the metric, as from equation 49.

Finding Λ
Contracting equation 57 in c and ρ as

$$\eta^{c\rho}\left(\partial_c\left(\partial_\rho f_\sigma - \partial_\sigma f_\rho\right) = \eta_{\sigma c}\partial_\rho\Lambda - \eta_{c\rho}\partial_\sigma\Lambda\right)$$

$$\implies \partial^\rho\left(\partial_\rho f_\sigma - \partial_\sigma f_\rho\right) = \delta^\rho{}_\sigma\partial_\rho\Lambda - n\partial_\sigma\Lambda$$

$$\implies \partial^\rho\partial_\rho f_\sigma - \partial^\rho\partial_\sigma f_\rho = \partial_\sigma\Lambda - n\partial_\sigma\Lambda$$

$$\implies \partial^2 f_\sigma - \partial_\sigma(\partial\cdot f) = (1-n)\partial_\sigma\Lambda$$

Now using $\Lambda = \dfrac{2}{n}(\partial\cdot f)$ we can write above as

$$\implies \partial^2 f_\sigma = (1-\frac{n}{2})\partial_\sigma\Lambda \tag{61}$$

Taking derivative of equation 61 we can write

$$\partial_\rho\partial^2 f_\sigma = (1-\frac{n}{2})\partial_\rho\partial_\sigma\Lambda$$

$$\partial^2\partial_\rho f_\sigma = (1-\frac{n}{2})\underbrace{\partial_\rho\partial_\sigma\Lambda}_{\text{symmetric}}$$

$$\implies \frac{1}{2}\partial^2(\partial_\rho f_\sigma + \partial_\sigma f_\rho) = (1-\frac{n}{2})\partial_\rho\partial_\sigma\Lambda$$

Using equation 54 in above we get

$$(2-n)\partial_\rho\partial_\sigma\Lambda = \eta_{\rho\sigma}\partial^2\Lambda \tag{62}$$

Now contracting equation 62 with $\eta^{\rho\sigma}$ as

$$(2-n)\eta^{\rho\sigma}\partial_\rho\partial_\sigma\Lambda = \eta^{\rho\sigma}\eta_{\rho\sigma}\partial^2\Lambda$$

$$\implies (2-n)\partial^2\Lambda = \partial^2\Lambda$$
$$\implies (n-1)\partial^2\Lambda = 0$$
$$\implies \partial^2\Lambda = 0 \quad for \quad n > 1 \tag{63}$$

Putting equation 63 in equation 62 we get

$$\partial_\rho\partial_\sigma\Lambda = 0 \quad for \quad n > 2 \tag{64}$$

Equation 64 suggests that Λ is at most linear in the coordinates, i.e;

$$\Lambda(x) = -2\alpha + 4c_a x^a \tag{65}$$

Putting equation 65 in equation 60 and doing the integration we find the infinitesimal transformation

$$f^a = a^a + \omega^{ba}x_b + \frac{1}{2}\int dx^a(-2\alpha + 4c_b x^b) + \frac{1}{2}\int dx^b \int \big(\partial_b(-2\alpha + 4c_\beta x^\beta)\ dx^a - \eta^{ab}\partial_b(-2\alpha + 4c_\beta x^\beta)\ dx_b\big)$$
$$= a^a + \omega^{ba}x_b - \alpha x^a + \frac{4}{2}c_b \int x^b dx^a + \frac{1}{2}\int dx^b \left(4c_b \int dx^a - 4\eta^{a\gamma}c_\gamma \int dx_b\right)$$
$$= a^a + \omega^{ba}x_b - \alpha x^a + 2c_b \int x^b dx^a + 2c_b \int x^a dx^b - 2\eta^{a\gamma}c_\gamma \int x_b dx^b$$
$$= a^a + \omega^{ba}x_b - \alpha x^a + 2c_b \int d(x^a x^b) - \eta^{ab}c_b\ x^2$$
$$\implies f^a = a^a + \omega^{ba}x_b - \alpha x^a + c_b\ (2\ x^a x^b - \eta^{ab}\ x^2) \tag{66}$$

In a n-dimensional space, first term gives n-translations, second term gives $\dfrac{n^2-n}{2}$ Lorentz transformations, third term gives one dilation(scaling) and the fourth term gives n- special conformal transformations. Hence, we have $n + \dfrac{n^2-n}{2} + 1 + n = \dfrac{(n+1)(n+2)}{2}$ parameters for a general transformation.

Note: for n=2 in equation 62 $\nRightarrow$ equation 64 which is crucial for finiteness of the conformal group, $\mathcal{C}(1, n-1)$. For n=2 every harmonic function Λ determines a solution $\Rightarrow \mathcal{C}(1,1)$ is infinite dimensional.

Conformal algebra

Defining δ_A to be the generators of the A-type coordinate transformation, where $A \equiv \{T, L, D, C\}$ and T := translation, L := Lorentz transformation, D := dilation and C := special conformal transformation. The conformal killing vector f can be written as

$$f^a = \lambda_A \delta_A^{(\lambda)} x^a \tag{67}$$
$$where \quad \lambda_A \equiv (a_a, \frac{1}{2}\omega_{ab}, \alpha, c_a)$$

where δ_A are defined as follows:

$$\delta_T^c\ x^a = \eta^{ca} \tag{68}$$
$$\implies f^a = a_c\ \delta_T^c\ x^a = a_c\ \eta^{ca} = a^a$$

$$\begin{aligned}
\delta_L^{ab}\ x^c &= \eta^{bc}x^a - \eta^{ac}x^b \\
\implies f^c = \frac{1}{2}\omega_{ab}\delta_L^{ab}\ x^c &= \frac{1}{2}\omega_{ab}(\eta^{bc}x^a - \eta^{ac}x^b) \\
&= \frac{1}{2}(\omega_{ab}\eta^{bc}x^a - \omega_{ab}\eta^{ac}x^b) \\
&= \frac{1}{2}(-\omega_{ba}\eta^{bc}x^a - \omega^c{}_b x^b) \\
&= \frac{1}{2}(-\omega^c{}_a x^a - \omega^c{}_b x^b) \\
&= \frac{1}{2}(-\omega^c{}_b - \omega^c{}_b)\ x^b \\
= -\omega^c{}_b\ x^b = -\omega^c{}_b\eta^{b\alpha}x_\alpha &= -\omega^{c\alpha}x_\alpha = \omega^{\alpha c}x_\alpha
\end{aligned} \tag{69}$$

$$\begin{aligned}
\delta_D\ x^a &= -x^a \\
\implies f^a = \alpha\delta_D\ x^a &= -\alpha x^a
\end{aligned} \tag{70}$$

$$\begin{aligned}
\delta_C^c\ x^a &= 2x^cx^a - \eta^{ca}x^2 \\
\implies f^a = c_c\delta_C^c\ x^a &= c_c(2x^cx^a - \eta^{ca}x^2)
\end{aligned} \tag{71}$$

From above we can infer the following:

$$\delta_T^a\ x^b \equiv \partial^a x^b = \eta^{a\beta}\partial_\beta x^b = \eta^{a\beta}\delta^b{}_\beta = \eta^{ab} \tag{72}$$

$$\delta_L^{ab}\ x^c = (x^a\partial^b - x^b\partial^a)x^c = \eta^{bc}x^a - \eta^{ac}x^b \tag{73}$$

$$\delta_D\ x^a = -x^\alpha\partial_\alpha x^a = -x^\alpha\delta^a{}_\alpha = -x^a \tag{74}$$

$$\delta_C^c\ x^a = (2x^cx^\alpha\partial_\alpha - x^2\partial^c)x^a = 2x^cx^a - \eta^{ca}x^2 \tag{75}$$

Now we can explicitly arrive at the conformal algebra:

$$\left[\delta_T^a, \delta_T^b\right] = \delta_T^a\ \delta_T^b - \delta_T^b\ \delta_T^a = \partial^a\ \partial^b - \partial^b\ \partial^a = 0 \tag{76}$$

$$\begin{aligned}
\left[\delta_L^{ab}, \delta_T^c\right] &= \delta_L^{ab}\ \delta_T^c - \delta_T^c\ \delta_L^{ab} \\
&= (x^a\partial^b - x^b\partial^a)\ \partial^c - \partial^c\ (x^a\partial^b - x^b\partial^a) \\
&= x^a\partial^b\partial^c - x^b\partial^a\partial^c - (\partial^c x^a)\partial^b - x^a\partial^c\partial^b + (\partial^c x^b)\partial^a + x^b\partial^c\partial^a \\
&= -(\partial^c x^a)\partial^b + (\partial^c x^b)\partial^a \\
\implies \left[\delta_L^{ab}, \delta_T^c\right] &= -\eta^{ca}\partial^b + \eta^{cb}\partial^a = \eta^{cb}\delta_T^a - \eta^{ca}\delta_T^b
\end{aligned} \tag{77}$$

$$\begin{aligned}
\left[\delta_L^{ab}, \delta_L^{cd}\right] &= \delta_L^{ab}\ \delta_L^{cd} - \delta_L^{cd}\ \delta_L^{ab} \\
&= (x^a\partial^b - x^b\partial^a)\ (x^c\partial^d - x^d\partial^c) - (x^c\partial^d - x^d\partial^c)\ (x^a\partial^b - x^b\partial^a) \\
&= x^a(\partial^b x^c)\partial^d + x^ax^c\partial^b\partial^d - x^a(\partial^b x^d)\partial^c - x^ax^d\partial^b\partial^c \\
&\quad -x^b(\partial^a x^c)\partial^d - x^bx^c\partial^a\partial^d + x^b(\partial^a x^d)\partial^c + x^bx^d\partial^a\partial^c \\
&\quad -x^c(\partial^d x^a)\partial^b - x^cx^a\partial^d\partial^b + x^c(\partial^d x^b)\partial^a + x^cx^b\partial^d\partial^a \\
&\quad +x^d(\partial^c x^a)\partial^b + x^dx^a\partial^c\partial^b - x^d(\partial^c x^b)\partial^a - x^dx^b\partial^c\partial^a \\
&= x^a(\partial^b x^c)\partial^d - x^a(\partial^b x^d)\partial^c - x^b(\partial^a x^c)\partial^d + x^b(\partial^a x^d)\partial^c
\end{aligned} \tag{78}$$

$$-x^c(\partial^d x^a)\partial^b + x^c(\partial^d x^b)\partial^a + x^d(\partial^c x^a)\partial^b - x^d(\partial^c x^b)\partial^a$$
$$= x^a\eta^{bc}\partial^d - x^a\eta^{bd}\partial^c - x^b\eta^{ac}\partial^d + x^b\eta^{ad}\partial^c$$
$$-x^c\eta^{da}\partial^b + x^c\eta^{db}\partial^a + x^d\eta^{ca}\partial^b - x^d\eta^{cb}\partial^a$$
$$= \eta^{bc}(x^a\partial^d - x^d\partial^a) + \eta^{bd}(x^c\partial^a - x^a\partial^c)$$
$$+\eta^{ac}(x^d\partial^b - x^b\partial^d) + \eta^{ad}(x^b\partial^c - x^c\partial^b)$$
$$\left[\delta_L^{ab}, \delta_L^{cd}\right] = \eta^{bc}\delta_L^{ad} + \eta^{bd}\delta_L^{ca} + \eta^{ac}\delta_L^{db} + \eta^{ad}\delta_L^{bc}$$

$$[\delta_D, \delta_T^a] = \delta_D\ \delta_T^a - \delta_T^a\ \delta_D \tag{79}$$
$$= -x^\alpha\partial_\alpha\partial^a + \partial^a(x^\alpha\partial_\alpha)$$
$$= -x^\alpha\partial_\alpha\partial^a + (\partial^a x^\alpha)\partial_\alpha + x^\alpha\partial^a\partial_\alpha = (\partial^a x^\alpha)\partial_\alpha$$
$$\implies [\delta_D, \delta_T^a] = \eta^{a\alpha}\partial_\alpha = \partial^a = \delta_T^a$$

$$[\delta_D, \delta_D] = \delta_D\ \delta_D - \delta_D\ \delta_D \tag{80}$$
$$\implies [\delta_D, \delta_D] = x^\alpha\partial_\alpha(x^\beta\partial_\beta) - x^\beta\partial_\beta(x^\alpha\partial_\alpha) = 0$$

$$\left[\delta_L^{ab}, \delta_D\right] = \delta_L^{ab}\ \delta_D - \delta_D\ \delta_L^{ab} \tag{81}$$
$$= -(x^a\partial^b - x^b\partial^a)\ (x^\gamma\partial_\gamma) +\ (x^\gamma\partial_\gamma)\ (x^a\partial^b - x^b\partial^a)$$
$$= -x^a(\partial^b x^\gamma)\partial_\gamma - x^a x^\gamma\partial^b\partial_\gamma + x^b(\partial^a x^\gamma)\partial_\gamma + x^b x^\gamma\partial^a\partial_\gamma$$
$$+x^\gamma(\partial_\gamma x^a)\partial^b + x^\gamma x^a\partial_\gamma\partial^b - x^\gamma(\partial_\gamma x^b)\partial^a - x^\gamma x^b\partial_\gamma\partial^a$$
$$\implies \left[\delta_L^{ab}, \delta_D\right] = 0$$

$$[\delta_D, \delta_C^a] = \delta_D\ \delta_C^a - \delta_C^a\ \delta_D \tag{82}$$
$$= (-x^\gamma\partial_\gamma)\ (2x^a x^\alpha\partial_\alpha - x^2\partial^a) + (2x^a x^\alpha\partial_\alpha - x^2\partial^a)\ (x^\gamma\partial_\gamma)$$
$$\implies [\delta_D, \delta_C^a] = -(2x^a x^\alpha\partial_\alpha - x^2\partial^a) = -\delta_C^a$$

$$\left[\delta_C^a, \delta_C^b\right] = \delta_C^a\ \delta_C^b - \delta_C^b\ \delta_C^a \tag{83}$$
$$= (2x^a x^\alpha\partial_\alpha - x^2\partial^a)\ (2x^b x^\beta\partial_\beta - x^2\partial^b)$$
$$-(2x^b x^\beta\partial_\beta - x^2\partial^b)\ (2x^a x^\alpha\partial_\alpha - x^2\partial^a)$$
$$\implies \left[\delta_C^a, \delta_C^b\right] = 0$$

$$\left[\delta_L^{ab}, \delta_C^c\right] = \delta_L^{ab}\ \delta_C^c - \delta_C^c\ \delta_L^{ab} \tag{84}$$
$$= (x^a\partial^b - x^b\partial^a)\ (2x^c x^\alpha\partial_\alpha - x^2\partial^c)$$
$$-(2x^c x^\alpha\partial_\alpha - x^2\partial^c)\ (x^a\partial^b - x^b\partial^a)$$
$$\implies \left[\delta_L^{ab}, \delta_C^c\right] = \eta^{bc}\delta_C^a - \eta^{ac}\delta_C^b$$

$$\left[\delta_C^a, \delta_T^b\right] = \delta_C^a\ \delta_T^b - \delta_T^b\ \delta_C^a \tag{85}$$
$$= (2x^a x^\alpha\partial_\alpha - x^2\partial^a)\ \partial^b - \partial^b\ (2x^a x^\alpha\partial_\alpha - x^2\partial^a)$$
$$= 2x^a x^\beta\partial_\beta\partial^b - x^2\partial^a\partial^b - 2\partial^b(x^a x^\beta)\partial_\beta$$
$$-2x^a x^\beta\partial^b\partial_\beta + \partial^b(x_\gamma x^\gamma)\partial^a + x^2\partial^b\partial^a$$
$$= -2\eta^{ba}x^\beta\partial_\beta + 2(x^b\partial^a - x^a\partial^b)$$
$$\implies \left[\delta_C^a, \delta_T^b\right] = 2\eta^{ba}\delta_D - 2\delta_L^{ab}$$

We summarise the Lie algebra of the conformal group from above:

$$\left[\delta_T^a, \delta_T^b\right] = 0 \tag{86}$$
$$\left[\delta_L^{ab}, \delta_T^c\right] = \eta^{cb}\delta_T^a - \eta^{ca}\delta_T^b \tag{87}$$
$$\left[\delta_L^{ab}, \delta_L^{cd}\right] = \eta^{bc}\delta_L^{ad} + \eta^{bd}\delta_L^{ca} + \eta^{ac}\delta_L^{db} + \eta^{ad}\delta_L^{bc} \tag{88}$$
$$\left[\delta_D, \delta_T^a\right] = \delta_T^a \tag{89}$$
$$\left[\delta_D, \delta_D\right] = 0 \tag{90}$$
$$\left[\delta_L^{ab}, \delta_D\right] = 0 \tag{91}$$
$$\left[\delta_D, \delta_C^a\right] = -\delta_C^a \tag{92}$$
$$\left[\delta_C^a, \delta_C^b\right] = 0 \tag{93}$$
$$\left[\delta_L^{ab}, \delta_C^c\right] = \eta^{bc}\delta_C^a - \eta^{ac}\delta_C^b \tag{94}$$
$$\left[\delta_C^a, \delta_T^b\right] = 2\eta^{ba}\delta_D - 2\delta_L^{ab} \tag{95}$$

Now by defining the basis as below

$$\iota p^a = \partial^a \tag{96}$$
$$\iota j^{ab} = (x^a\partial^b - x^b\partial^a) \tag{97}$$
$$\iota d = -x^\alpha\partial_\alpha \tag{98}$$
$$\iota k^a = 2x^a x^\beta\partial_\beta - x^2\partial^a \tag{99}$$

With these notations for the generators, we can write the Lie algebra(86-95)

$$\left[p^a, p^b\right] = 0 \tag{100}$$
$$\iota\left[j^{ab}, p^c\right] = \eta^{ca}p^b - \eta^{cb}p^a \tag{101}$$
$$\iota\left[j^{ab}, j^{cd}\right] = \eta^{bc}j^{ad} - \eta^{bd}j^{ac} - \eta^{ac}j^{bd} + \eta^{ad}j^{bc} \tag{102}$$
$$\iota\left[d, p\right] = p \tag{103}$$
$$\left[d, d\right] = 0 \tag{104}$$
$$\left[j^{ab}, d\right] = 0 \tag{105}$$
$$\iota\left[d, k\right] = -k \tag{106}$$
$$\left[k^a, k^b\right] = 0 \tag{107}$$
$$\iota\left[j^{ab}, k^c\right] = \eta^{ac}k^b - \eta^{bc}k^a \tag{108}$$
$$\iota\left[k^a, p^b\right] = 2\eta^{ab}d - 2j^{ab} \tag{109}$$

Next we consider various subgroups of the conformal group and the corresponding Lie subalgebra and obtain unitary representation

Lie subalgebras

Subgroup: Scale + Translation

We consider the transformation

$$\tilde{x} = T(\alpha, a)x = e^\alpha\ x + a \tag{110}$$

Next we consider two such transformations

$$
\begin{aligned}
T(\alpha,a)T(\beta,b)x &= T(\alpha,a)(e^{\beta}\ x+b)\\
&= e^{\alpha}\ (e^{\beta}\ x+b)+a\\
&= e^{\alpha+\beta}\ x+be^{\alpha}+a\\
\implies T(\alpha,a)T(\beta,b) &= T(\alpha+\beta,a+be^{\alpha}) \qquad (111)
\end{aligned}
$$

Following the same line of arguments we can show that

$$\{T(\alpha,a)T(\beta,b)\}T(\gamma,c) \equiv T(\alpha,a)\{T(\beta,b)T(\gamma,c)\} = T(\alpha+\beta+\gamma,a+be^{\alpha}+ce^{\alpha+\beta}) \qquad (112)$$

Using the multiplication law(equation 111) we can show that

$$
\begin{aligned}
T(\alpha,a)T(0,0) &= T(\alpha,a) \qquad (113)\\
\implies T(0,0)\ &is\ the\ Identity\ element\\
T(\alpha,a)T(-\alpha,-ae^{-\alpha}) &= T(\alpha-\alpha,a+(-ae^{-\alpha})e^{\alpha})\\
&= T(0,0)=T(-\alpha,-ae^{-\alpha})T(\alpha,a)\\
T^{-}(\alpha,a) &= T(-\alpha,-ae^{-\alpha}) \qquad (114)\\
\implies\ &An\ Inverse\ exists
\end{aligned}
$$

Therefore, $\{T(\alpha,a)\}$ forms a group with the multiplication law as in equation 111. These coordinate transformations induce unitary linear transformations on the vectors(quantum states) in the physical Hilbert space. These form a unitary representation of the group.

$$\mathcal{U}(\alpha,a)\mathcal{U}(\beta,b) = \mathcal{U}(\alpha+\beta,a+be^{\alpha}) \qquad (115)$$

$$\mathcal{U}(0,0) = \mathcal{U}(E)\ Identity\ operator \qquad (116)$$

$$\mathcal{U}^{-}(\alpha,a) = \mathcal{U}(-\alpha,-ae^{-\alpha})\ Inverse \qquad (117)$$

For infinitesimal coordinate transformations, $\mathcal{U}$ with parameters (α,a) can be written as

$$\mathcal{U}(\alpha,a) \approx 1+\iota\alpha D+\iota a^{a}P_{a} \qquad (118)$$

Since $\mathcal{U}$ is unitary; (D,P) are Hermitian.

Transformation laws of D and P

Similar to finding the transformation laws of the generators of the Poincaré algebra we can find the transformation laws of D and P by first showing

$$\mathcal{U}(\beta,b)\mathcal{U}(\alpha,a)\mathcal{U}^{-1}(\beta,b) = \mathcal{U}(\alpha,ae^{\beta}-\alpha b) \qquad (119)$$

using the composition rule in relation 111 and considering (α,a) to infinitesimal. Here (β, b) are parameters of the new transformation unrelated to (α,a). Next, for infinitesimal (α,a)

transformation we can write relation 119 as

$$\begin{aligned}
\mathcal{U}(\beta,b)(1+\iota\alpha D+\iota a^aP_a)\mathcal{U}^{-1}(\beta,a) &= \mathcal{U}(\alpha, ae^{\beta}-\alpha b)\\
L.H.S = 1+\iota\alpha\mathcal{U}(\beta,b)D\mathcal{U}^{-1}(\beta,a)+\iota a^a\mathcal{U}(\beta,b)P_a\mathcal{U}^{-1}(\beta,a)&\\
R.H.S &= \mathcal{U}(\alpha, ae^{\beta}-\alpha b)\\
&= 1+\iota\alpha D+\iota(ae^{\beta}-\alpha b)^aP_a\\
&= 1+\iota\alpha(D-b^aP_a)+\iota a^ae^{\beta}P_a\\
Now\ equating\ coefficients\ of\ \alpha\ and\ a^a\ on\ both\ sides&\\
\implies \mathcal{U}D\mathcal{U}^{-1} &= D-b^aP_a\\
\mathcal{U}^{-1}D\mathcal{U} &= D+b^a\mathcal{U}^{-1}P_a\mathcal{U}\\
and \implies \mathcal{U}P_a\mathcal{U}^{-1} &= e^{\beta}P_a\\
\mathcal{U}^{-1}P_a\mathcal{U} &= e^{-\beta}P_a \equiv \bar{P}_a \qquad (120)\\
\implies \bar{D}\equiv\mathcal{U}^{-1}D\mathcal{U} &= D+b_aP^ae^{-\beta} \qquad (121)
\end{aligned}$$

Next we consider (β,b) to be infinitesimal and use

$$\mathcal{U}^{-1}(\beta,b) = \mathcal{U}(-\beta,-be^{-\beta}) \approx \mathcal{U}(-\beta,-b(1-\beta)) = \mathcal{U}(-\beta,-b) \qquad (122)$$

Now from relation 120 and 121 we can write

$$\begin{aligned}
(1-\iota\beta D+\iota b_lP^l)P^a(1+\iota\beta D+\iota b_lP^l) &= (1-\beta+O[\beta^2])P^a\\
Equating\ coefficients\ of\ \beta\ on\ both\ sides\ we\ get&\\
\iota DP^a-P^a\iota D &= P^a\\
\implies [\iota D,P^a] &= P^a \qquad (123)\\
Also, equating\ coefficients\ of\ b_l\ on\ both\ sides\ we\ get&\\
-\iota P^lP^a+P^a\iota P^l &= 0\\
\implies \left[\iota P^l,P^a\right] &= 0 \qquad (124)\\
Similarly\ considering\ the\ other\ relation\ we\ get&\\
[\iota D,D] &= 0 \qquad (125)
\end{aligned}$$

Commutators 123-125 form the Lie subalgebra of the group of scale and translation.

Subgroup: Poincaré group

This has been worked out in detail on page 10-12.

Subgroup: Scale + Lorentz

We consider here the following transformation

$$\tilde{x} = T(\alpha,\Lambda)x = e^{\alpha}\ \Lambda\ x \qquad (126)$$

Next, consider two such transformations to get the multiplication law of the subgroup

$$\begin{aligned}
T(\alpha,\Lambda)T(\beta,\bar{\Lambda})x &= T(\alpha,\Lambda)(e^{\beta}\ \bar{\Lambda}\ x)\\
&= e^{\alpha}\Lambda(e^{\beta}\ \bar{\Lambda}\ x)\\
&= e^{\alpha+\beta}\ \Lambda\ \bar{\Lambda}\ x\\
\implies T(\alpha,\Lambda)T(\beta,\bar{\Lambda}) &= T(\alpha+\beta,\Lambda\bar{\Lambda}) \qquad (127)
\end{aligned}$$

The unitary transformations have the same composition rule

$$\mathcal{U}(\alpha, \Lambda)\mathcal{U}(\beta, \bar{\Lambda}) = \mathcal{U}(\alpha + \beta, \Lambda\bar{\Lambda}) \tag{128}$$

From the composition rule(128) we see that

$$\mathcal{U}(\alpha, 1)\mathcal{U}(0, \Lambda)\mathcal{U}(-\alpha, 1) = \mathcal{U}(0, \Lambda) \tag{129}$$

Next, for infinitesimal (α, Λ) we can write

$$\mathcal{U}(\alpha, 1) = 1 + \iota\alpha D \tag{130}$$
$$\mathcal{U}(0, \Lambda) = 1 + \frac{\iota}{2}\,\omega \cdot J \tag{131}$$
$$\mathcal{U}^{-1}(\alpha, 1) = 1 - \iota\alpha D \tag{132}$$

Using the infinitesimal expansions for $\mathcal{U}(0, \Lambda)$ in relation 129 we can write

$$\begin{aligned} \mathcal{U}(\alpha, 1)(1 + \frac{\iota}{2}\,\omega_{ab}J^{ab})\mathcal{U}(-\alpha, 1) &= 1 + \frac{\iota}{2}\,\omega_{ab}J^{ab} \\ \implies \mathcal{U}(\alpha, 1)J^{ab}\mathcal{U}(-\alpha, 1) &= J^{ab} \end{aligned} \tag{133}$$

Next we use the expansions 130 and 132 in relation 133

$$\begin{aligned} &\mathcal{U}(\alpha, 1)J^{ab}\mathcal{U}(-\alpha, 1) \\ &= (1 + \iota\alpha D)J^{ab}(1 - \iota\alpha D) = J^{ab} \\ &\implies \left[\iota J^{ab}, D\right] = 0 \end{aligned} \tag{134}$$

Next we postulate that K^c is a Lorentz vector and that the algebra is complete with generators P^a, J^{ab}, D and K^c

$$\left[\iota J^{ab}, K^c\right] = \eta^{ac}K^b - \eta^{bc}K^a \tag{135}$$

With this postulate and the subalgebras derived in the previous sections we can derive the complete algebra of the group using Jacobi Identities.

We have

$$[\iota D, K^a] = aK^a + bP^a \tag{136}$$
$$\left[\iota K^a, K^b\right] = cJ^{ab} \tag{137}$$
$$\left[\iota K^a, P^b\right] = d\eta^{ab}D - dJ^{ab} \tag{138}$$

Using (136) we can verify the Jacobi Identities with J :

$$\begin{aligned} &\left[\iota J^{ab}, [\iota D, K^c]\right] + \left[\iota D, \left[\iota K^c, J^{ab}\right]\right] + \left[\iota K^c, \left[\iota J^{ab}, D\right]\right] \\ &= \left[\iota J^{ab}, aK^c + bP^c\right] + \left[\iota D, \eta^{bc}K^a - \eta^{ac}K^b\right] + [\iota K^c, 0] \\ &= a\left[\iota J^{ab}, K^c\right] + b\left[\iota J^{ab}, P^c\right] + \eta^{bc}\left[\iota D, K^a\right] - \eta^{ac}\left[\iota D, K^b\right] \\ &= a\,(\eta^{ac}K^b - \eta^{bc}K^a) + b\,(\eta^{ac}P^b - \eta^{bc}P^a) \\ &+\eta^{bc}\,(aK^a + bP^a) - \eta^{ac}\,(aK^b + bP^b) = 0 \\ \implies &\left[\iota J^{ab}, [\iota D, K^c]\right] + \left[\iota D, \left[\iota K^c, J^{ab}\right]\right] + \left[\iota K^c, \left[\iota J^{ab}, D\right]\right] = 0 \end{aligned} \tag{139}$$

Similarly the others can be verified. Next we use the Jacobi Identities to find the values for the constants a,b,c and d.

$$\begin{aligned}
\left[\iota D, \left[\iota K^a, P^b\right]\right] + \left[\iota K^a, \left[\iota P^b, D\right]\right] + \left[\iota P^b, \left[\iota D, K^a\right]\right] \\
= \left[\iota D, d\eta^{ab} D - dJ^{ab}\right] + \left[\iota K^a, -P^b\right] + \left[\iota P^b, aK^a + bP^a\right] \\
= d\eta^{ab}\left[\iota D, D\right] - d\left[\iota D, J^{ab}\right] + \left[\iota K^a, -P^b\right] + a\left[\iota P^b, K^a\right] + b\left[\iota P^b, P^a\right] \\
= d\eta^{ab}(0) - d(0) - \left[\iota K^a, P^b\right] - a\left[\iota K^a, P^b\right] + b(0) \\
= (-1-a)\ \left[\iota K^a, P^b\right] = 0 \\
\implies a = -1
\end{aligned} \tag{140}$$

Similarly using the following Jacobi Identities we can infer that c = bd = 0

$$\left[\iota D, \left[\iota K^a, K^b\right]\right] + \left[\iota K^a, \left[\iota K^b, D\right]\right] + \left[\iota K^b, \left[\iota D, K^a\right]\right] = 0 \tag{141}$$

$$\left[\iota P^a, \left[\iota K^b, K^c\right]\right] + \left[\iota K^b, \left[\iota K^c, P^a\right]\right] + \left[\iota K^c, \left[\iota P^a, K^b\right]\right] = 0 \tag{142}$$

Now, the choice b = 0 and d = 2 is compatible with the previous calculations. Even $\left[\iota K^a, P^b\right] = 0$ is compatible with the Jacobi Identities but then we will have an arbitrary real factor in $[\iota D, K^a]$. Therefore to summarise,

$$[\iota D, K^a] = -K^a \tag{143}$$

$$\left[\iota K^a, K^b\right] = 0 \tag{144}$$

$$\left[\iota K^a, P^b\right] = 2\eta^{ab} D - 2J^{ab} \tag{145}$$

Isomorphism to $\mathcal{SO}$(2,n)

$\mathcal{SO}$(2,n) is the "Lorentz" group in a (n+2)-dimensional flat spacetime. It can be considered as the set of pseudo-orthogonal transformations in (n+2)-dimensional flat spacetime. The metric is identified as

$$\eta^{\mu\nu} = (-1, 1, \eta^{ab}) \tag{146}$$

Where $(\mu, \nu) = -2, -1, 0, 1, \ldots, n-1$ and $(a, b) = 0, 1, \ldots, n-1$. The Lie algebra of $\mathcal{SO}$(2,n) is as before,

$$[\iota M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} \tag{147}$$

Both $\mathcal{SO}$(2,n) and $\mathcal{C}$(1,n-1) have the same number of generators $(= \dfrac{(n+1)(n+2)}{2})$. Now, by defining the following generators we can recover the conformal algebra

$$M^{-2,-1} = D \tag{148}$$

$$M^{ab} = J^{ab} \tag{149}$$

$$M^{-2,a} = \frac{1}{2}(P^a - K^a) \tag{150}$$

$$M^{-1,a} = \frac{1}{2}(P^a + K^a) \tag{151}$$

Hence, the isomorphism of $\mathcal{C}$(1,n-1) to $\mathcal{SO}$(2,n) is established.

Conformal invariance in nature

We know that Poincaré invariance is realised in nature (special relativistic laws based on the invariance hold for all inertial observers). However, conformal invariance(for $n > 2$) is not a symmetry of nature. To see that, consider

$$[\iota D, P^a] = P^a \tag{152}$$

the commutator of conformal algebra. Consider

$$\begin{aligned} O(\alpha) &= e^{-\iota\alpha D}\ P^2\ e^{\iota\alpha D} \\ \frac{dO(\alpha)}{d\alpha} &= e^{-\iota\alpha D}\ \left[\iota D, P^2\right]\ e^{\iota\alpha D} \\ &= -2e^{-\iota\alpha D}\ P^2\ e^{\iota\alpha D} \\ &= -2O(\alpha) \\ O(\alpha) &= e^{-2\alpha}\ O(0) = e^{-2\alpha} P^2 \\ \implies\ e^{-\iota\alpha D}\ P^2\ e^{\iota\alpha D} &= e^{-2\alpha} P^2 \end{aligned} \tag{153}$$

Now, if $|P>$ is some one-particle state with mass m

$$P^2|P>= m^2|P>$$

then the state $|\bar{P}>$ has a mass given by $\bar{m}^2 = e^{2\alpha}m^2$

$$\begin{aligned} where\ \ |\bar{P}> &= e^{-\iota\alpha D}|P> \qquad (154) \\ P^2|\bar{P}> &= P^2 e^{-\iota\alpha D}|P> \\ &= e^{2\alpha}e^{-\iota\alpha D}P^2|P> \quad (using\ 153) \\ &= e^{2\alpha}m^2 e^{-\iota\alpha D}|P>= e^{2\alpha}m^2|\bar{P}> \\ That\ is\ \ P^2|\bar{P}> &= e^{2\alpha}m^2|\bar{P}> \qquad (155) \end{aligned}$$

This means $P^a|\bar{P}>= e^{\alpha}\ |\bar{P}>$. If the vacuum is invariant under the scale transformation i.e; $e^{-\iota\alpha D}|0>= |0>$, there is a single vacuum state from which all states must be constructed. Thus, $|P>= a^{\dagger}(P)|0>$ and $|\bar{P}>= a^{\dagger}(e^{\alpha}P)|0>$. In other words, conformal invariance requires that for each momentum state $|P>$, there exists a rescaled momentum state $|\bar{P}>= |e^{\alpha}P>$. This means that for each state of mass m, there exist states with mass given by $e^{\alpha}m$. Since α is an arbitrary real constant, conformal invariance implies that either all states have mass zero or that there is a continuous mass spectrum. Both implications are physically unacceptable. Hence, conformal symmetry is not thought to be relevant to the physical world.

In this report we have not considered the n =2 case(for which conformal algebra is infinite dimensional). Conformal symmetries for n=2 case is considered in some advanced theories(like string theory) that is beyond the scope of this project. For $n > 2$, though conformal invariance is not satisfied by nature, it is possible that broken conformal invariance (broken spontaneously or otherwise) may have some significance.

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